First $\lim_{n \doteq \infty} \frac{1}{\kappa n^{\kappa+1}} S_{n,\kappa}$ must be sought.

Only in the last term of $S_{n,\kappa}$ n appears in $\kappa+1$ factors, therefore the preceding terms disappear, and

$$\lim_{n \doteq \infty} \frac{1}{\kappa n^{\kappa+1}} S_{n,\kappa} = \lim_{n \doteq \infty} \frac{1}{\kappa n^{\kappa+1}} \left[\frac{n(n-1)(n-2).....(n-\kappa)}{(\kappa+1)!} \kappa! \right] = \frac{1}{\kappa(\kappa+1)}.$$

Therefore $\lim_{n \doteq \infty} \sum_{\lambda=1}^{\lambda=n} \sum_{\kappa=1}^{\kappa=n} (-1)^{\kappa-1} \frac{\lambda^\kappa}{\kappa n^{\kappa+1}} = \sum_{\kappa=1}^{\kappa=\infty} (-1)^{\kappa-1} \frac{1}{\kappa(\kappa+1)}$

$$= \frac{1}{1.2} - \frac{1}{2.3} + \frac{1}{3.4} - \frac{1}{4.5} + ... = (1-\frac{1}{2}) - (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) - .....$$

$$= 2(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + ...) - 1 = 2 \log 2 - 1 = \log \frac{4}{e}.$$

Hence, $\log x = \log \frac{4}{e}$; and $x = \lim_{n \doteq \infty} \frac{1}{n} \sqrt[n]{[(n+1)(n+2) .... 2n]} = \frac{4}{e}$.

II. **Solution by S. A. COREY, Hiteman, Iowa.**

Evidently, $\frac{1}{n} \sqrt[n]{[(n+1)(n+2) .... (2n)]}$

$$= \frac{1}{n} \sqrt[n]{n^n (1+\frac{1}{n})(1+\frac{2}{n}) .... (2)}$$

$$= \sqrt[n]{(1+\frac{1}{n})(1+\frac{2}{n}) ....(2)} = s \text{ (say)}.$$

Therefore, $\log s = \frac{1}{n} [\log(1+\frac{1}{n}) + \log(1+\frac{2}{n}) + ... + \log 2]$.

Letting $dx = 1/n$, we have,

$$\lim_{n \doteq \infty} \log s = \int_1^2 \log x \, dx = 2 \log 2 - 1, \text{ or } s = \frac{4}{e}.$$

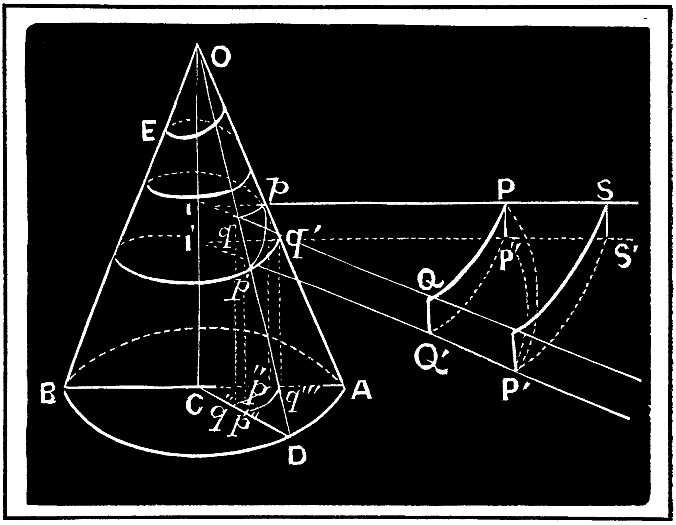
Also solved by Henry Heaton, and J. Scheffer. Several incorrect solutions were received.

209. **Proposed by J. EDWARD SANDERS, Reinersville, Ohio.**

A thread makes $n(=30)$ equidistant spiral turns around a rough cone whose altitude is $h(=10$ feet), and radius of base $r(=11$ inches). How far will a bird fly in unwinding the thread if the part unwound is at all times perpendicular to the axis of the cone?

Solution by Professor B. F. FINKEL, A. M., 4038 Locust Street, Philadelphia, Pa.

Let P and P' be two consecutive positions of the bird at any time; p and p' the two corresponding positions of the end of the string in contact with the cone, it being assumed that the string adheres slightly to the cone in order that the conditions of the problem be fulfilled; $s=pP=OEP$, the length of the string unwound at any time; $Op=w$, $pq'=dw$; $pp'=ds$; $\theta=$ the angle between Ip and the line perpendicular to OC at the beginning of the flight, the angle being measured in the direction in which the bird flys around the cone; $d\theta=$the angle $pIq=$the angle ACD; $s_1=$the distance the bird has flown at any time; $R=$the radius of the base of the cone; and $l=$its slant height. Then



$$ds=[(pq')^2+(q'p')^2]^{\frac{1}{2}}=[(\frac{Rw}{l}d\theta)^2+dw^2]^{\frac{1}{2}} \text{ ....(1).}$$

Since the string passes around the cone n times, it follows that

$$w : R\theta=\frac{l}{n} : 2\pi R, \text{ or } w\frac{l\theta}{2\pi n}, \text{ and } dw=\frac{ld\theta}{2\pi n} \text{ ....(2).}$$

Hence, $ds=\frac{1}{2\pi n}[R^2\theta^2+l^2]d\theta$......... (3), and

$$s=\frac{1}{2\pi n}\int_0^\theta \sqrt{[R^2\theta^2+l^2]}d\theta=\frac{R}{4\pi n}[\theta\sqrt{(k^2+\theta^2)}+k^2\log(\frac{\theta+\sqrt{(k^2+\theta^2)}}{k})] \text{ .... (4),}$$

where $k=l/R$. If $\theta=2\pi n$, this expression gives the complete length of the string.

Now, $ds_1=PP'=[P'Q'^2+Q'P''^2+QQ'^2]^{\frac{1}{2}}$ ....(5).

But $P''Q'=PQ=PId\theta=(Pp+pI)d\theta=(s+\frac{R\theta}{2\pi n})d\theta$; $Q'P'=pp'=ds$

$$=\frac{R}{2\pi n}\sqrt{(k^2+\theta^2)}d\theta\text{; and } QQ'=II'=pq'\cos\angle AOC=\frac{\sqrt{(l^2-R^2)}}{l}dw$$

$$=\frac{1}{2\pi n}\sqrt{(l^2-R^2)}d\theta.$$

Hence, by substituting these values of $P'Q'$, $Q'P''$, and QQ' and the value of s from (4) in (5), we have

$$ds_1=\frac{R}{4\pi n}\{[\theta\sqrt{(k^2+\theta^2)}+k^2\log(\frac{\theta+\sqrt{(k^2+\theta^2)}}{k})+2\theta]^2+4\theta^4+8k^2-4\}^{\frac{1}{2}}\,d\theta.$$

$$\text{Hence, } s_1=\frac{R}{4\pi n}\int_0^{2\pi n}\{[\theta\sqrt{(k^2+\theta^2)}+k^2\log(\frac{\theta+\sqrt{(k^2+\theta^2)}}{k})$$

$$+2\theta]^2+4\theta^2+8k^2-4\}^{\frac{1}{2}}\,d\theta\text{.........(6)}.$$

$$\text{Let } \theta=\tfrac{1}{2}k\left\{\left[\frac{2\pi n+\sqrt{(k^2+4\pi^2n^2)}}{k}\right]^x-\left[\frac{2\pi n+\sqrt{(k^2+4\pi^2n^2)}}{k}\right]^{-x}\right\}$$

$$=\tfrac{1}{2}k[a^x-a^{-x}]=k\sinh(x\log a),\text{ where } a=\frac{2\pi n+\sqrt{(k^2+4\pi^2n^2)}}{k}$$

Hence, when $\theta=0$, $x=0$, and when $\theta=2\pi n$, $x=1$.

$$\text{Then } s_1=\frac{l}{4\pi n}\log a\int_0^1\{[\tfrac{1}{2}k^2\sinh 2(x\log a)+k^2\log a^x+2k\sinh(x\log a)]^2$$

$$+4k^2\sinh^2(x\log a)+8k^2-4\}^{\frac{1}{2}}\cosh(x\log a)dx.$$

If we divide the interval (0, 1) into 10 equal parts and find the value of

$$\{[\tfrac{1}{2}k^2\sinh 2(x\log a)+k^2x\log a+2k\sinh(x\log a)]^2$$
$$+4k^2\sinh^2(x\log a)+8k^2-4\}^{\frac{1}{2}}\cosh(x\log a)$$

for each of the values of $x=0, .1, .2, \ldots\ldots, .9, 1$ and if these values be designated by $A_0, A_1, A_2, \ldots\ldots, A_{10}$, respectively, we have, by Cotes' Method of Approximate Quadrature,*

$$s_1=\frac{l}{4\pi n}\log\left(\frac{2\pi n+\sqrt{(k^2+4\pi^2n^2)}}{k}\right)$$

$$\times\frac{16067(A_0+A_{10})+106300(A_1+A_9)-48525(A_2+A_8)+272400(A_3+A_7)}{598752}$$

$$+\frac{-260550(A_4+A_6)+427368A_5}{598752}.$$

*See Roger Cotes' *Opera Miscellanea*, p. 33.

The ordinates A_0, A_1, ..., A_{10} may be easily computed by means of a table of Hyperbolic Functions.

In this solution it has been assumed that the unwinding of the string begins at the vertex of the cone. If the unwinding begins at the base of the cone, we replace, in (3), θ by $(2\pi n-\theta)$ and take the negative sign of the radical since it is then a decreasing function of θ. This gives

$$s=\frac{R}{4\pi n}\Big[2\pi n\sqrt{(4\pi^2n^2+k^2)}-(2\pi n-\theta)\sqrt{[(2\pi n-\theta)^2+k^2]}$$

$$+k^2\log\Big(\frac{2\pi n+\sqrt{(4\pi^2n^2+k^2)}}{2\pi n-\theta+\sqrt{[(2\pi n-\theta)^2+k^2]}}\Big)\Big].$$

We then have, $ds_1=[(s+\frac{(2\pi n-\theta)Rd\theta}{2\pi n})^2+ds^2+\frac{l^2-R^2}{4\pi^2n^2}d\theta^2]^{\frac{1}{2}}$ ....(7).

But $ds=-\frac{R}{2\pi n}\sqrt{[(2\pi n-\theta)^2+k^2]}d\theta$.

Substituting the values of s and ds in (7) and letting $LR/4\pi n$=the entire length of the string, and $\theta=2\pi n-k\sinh[(1-x)\log a]$, where

$$a=\frac{2\pi n+\sqrt{(4\pi^2n^2+k^2)}}{k},$$

we have,

$$s_1=\frac{R}{4\pi n}\int_0^1\Big[\{L-\tfrac{1}{2}k^2\sinh[2(1-x)\log a]+k^2(1-x)\log a$$

$$+2k\sinh[(1-x]\log a]\}^2+4k^2\sinh^2[(1-x)\log a]$$

$$+8k^2-4\Big]^{\frac{1}{2}}\cosh[(1-x)\log a]dx,$$

the value of which may be obtained by the foregoing method of approximation.

DIOPHANTINE ANALYSIS.

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126. Proposed by R. A. THOMPSON, M. A., C. E., Engineer Railroad Commission of Texas.

Eight persons wish to play a series of games of progressive duplicate whist. In one evening, 12 boards are played, 4 boards (and return) by one couple against each of the other three couples, the same partners being retained throughout one evening. How many evenings will be required to complete the series, and what is the order of play, it being required that each player shall play with every other player as partner, and that each couple shall play once and but once against every other couple.